FTD-TT-63-283

63-3-2

TRANSLATION

SOLVING THE BOUNDARY LAYER PROBLEM

BY

V. Ya. Shkadov

FOREIGN TECHNOLOGY DIVISION

AIR FORCE SYSTEMS COMMAND

WRIGHT-PATTERSON AIR FORCE BASE

OHIO

UNEDITED ROUGH DRAFT TRANSLATION

SOLVING THE BOUNDARY LAYER PROBLEM

By: V. Ya. Shkadov

English Pages: 6

Source: Russian Periodical, Izvestiya Adademii Nauk SSSR, Otdeleniye Tekhnicheskikh Nauk, Mekhanika i Mashinostroyeniye, No. 3, 1962, pp. 173-175.

SOV/179-62-0-3

SOLVING THE BOUNDARY LAYER PROBLEM

V. Ya. Shkadov

The movement of an incompressible viscous fluid in a boundary layer is described in Kotschin et al. [1] by the equation for the stream function ψ (x,y)

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=U\frac{dU}{dx}+\nu\frac{\partial^3\psi}{\partial y^3} \tag{1}$$

with the boundary conditions

$$\psi=0,\quad \frac{\partial\psi}{\partial y}=0 \text{ when } y=0,\qquad \frac{\partial\psi}{\partial y}\to U \text{ when } y\to\infty \tag{2}$$

If the velocity of the fluid U(x) on the external boundary of the boundary layer is given in the form of a power series $U(x) = a_0 + a_1x + a_2x^2 + \ldots$, the solution for ψ (x,y) may also be found in the form of a series with respect to x, the coefficients of which must be found by numerical integration.

At present Curle [2] has carried calculations out to the x^{11} term for bodies with blunted fronts and symmetrical with respect to the onflowing stream. When using this solution one must limit himself to the first few terms of the U(x) expansion, but this is not always sufficient, especially for asymmetrical bodies. In these cases one

must use a solution which cannot represent U(x) as a rapidly converging series.

The author himself [3] has pointed out the possibility of producing a solution to the boundary layer equations definable by the dimensionless combinations $U'x/U$, $U''x^2/U$, $U'''x^3/U$, ..., in which the primes indicate differentiation with respect to $\underline{x}$.

We will introduce a new independent variable $\eta = \eta\ (x,y)$ and the unknown function f (x,η) so that

$$\psi(x, y) = \sqrt{U\nu x}\, f(x, \eta) \qquad \left(\eta = y\sqrt{\frac{U}{\nu x}}\right)$$

For f (x,η) is derived the equation

$$\frac{\partial^3 f}{\partial\eta^3} + \frac{1}{2} f \frac{\partial^2 f}{\partial\eta^2} + \frac{U'x}{U}\left[1 - \left(\frac{\partial f}{\partial\eta}\right)^2 + \frac{1}{2} f \frac{\partial^2 f}{\partial\eta^2}\right] = x\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial x\,\partial\eta} - \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial\eta^2}\right) \qquad (3)$$

with the bounding conditions

$$f = 0, \quad \frac{\partial f}{\partial\eta} = 0 \text{ when } \eta = 0, \qquad \frac{\partial f}{\partial\eta} \to 1 \text{ when } \eta \to \infty \qquad (4)$$

Let us examine the equation

$$\frac{\partial^3 f}{\partial\eta^3} + \frac{1}{2} f \frac{\partial^2 f}{\partial\eta^2} + \frac{U'x}{U}\left[1 - \left(\frac{\partial f}{\partial\eta}\right)^2 + \frac{1}{2} f \frac{\partial^2 f}{\partial\eta^2}\right] = \\ = \sum_{i=1}^{\infty} (ip_i - p_1 p_i + p_{i+1})\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial p_i\,\partial\eta} - \frac{\partial f}{\partial p_i}\frac{\partial^2 f}{\partial\eta^2}\right) \qquad (5)$$

in which the function depends on η, p_1, p_2, ...

We will look for a solution to this equation which satisfies the boundary conditions (4). If, in the space of the variables p_1, p_2, ..., function U(x) parametrically gives a curve with the relationships $p_1 = U'x/U$, $p_2 = U''x^2/U$, it is easy to see that along this curve

$$x\frac{dp_i}{dx} = ip_i - p_1 p_i + p_{i+1} \qquad (i = 1, 2, \ldots)$$

therefore the right half of Eq. (5) represents

$$x\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial x\,\partial\eta} - \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial\eta^2}\right)$$

and Eq. (5) coincides with (3).

Consequently, in this special case function $f(\eta, p_1, p_2, \ldots)$ found by the solution of Eq. (5) with boundary conditions (4) will describe the flow in the boundary layer.

Two solutions for Eq. (5) may be produced: expansion into a series with respect to $p_1, p_2, \ldots$ and into one with respect to $p_1-1, p_2, \ldots$ The first solution corresponds to boundary layer flow beginning at the sharp edge on which $p_1 = 0$, $p_2 = 0, \ldots$, while the second describe a boundary layer beginning at a critical point at which $p_1 = 1$, $p_2 = 0, \ldots$

The coefficients of these series are functions of η and are found by numerical integration of ordinary differential equations. The necessary calculations were done on a "Strela" computer. They show that the coefficients of the series rapidly diminish as i grows, therefore in what follows the terms containing $p_4, p_5, \ldots$ are considered too small and are dropped. For bodies with a sharp forward edge

$$f = f_{00} + \delta f_{10}p_1 + \delta^2 (f_{20}p_1^2 + f_{21}p_2) + \delta^3 (f_{30}p_1^3 + f_{31}p_1p_2 + f_{32}p_3) + \cdots \tag{6}$$

Substituting f into Eq. (5) and setting expressions with various combinations of p_1, p_2, and p_3 equal to zero we may derive equations for f_{ik}.

Because, however, numerical integration necessitates awkward right sides and consequent storage of much information, f_{ik} was computed by a different method. Let

$$U(x) = 1 + ax + bx^2 + cx^3 \qquad (a, b, c = \text{const})$$

It can be found that

$$ax = \frac{1}{d}\left(p_1 - \frac{1}{2}p_2 + \frac{1}{6}p_3\right), \qquad bx^2 = \frac{1}{d}\left(\frac{1}{2}p_2 - \frac{1}{2}p_3\right), \tag{7}$$
$$cx^3 = \frac{1}{d}\left(\frac{1}{6}p_3\right), \qquad d = 1 - p_1 + \frac{1}{2}p_2 - \frac{1}{6}p_3.$$

By expanding into a series with respect to $\underline{x}$ we will find the solution to Eq. (3) with the boundary conditions (4) in the form

$$f = f_0 + a g_1 x + (a^2 h_1 + b h_2) x^2 + (a^3 k_1 - a b k_2 + c k_3) x^3 + \cdots \tag{8}$$

The functions $f_0(\eta)$, $g_1(\eta)$, $h_1(\eta)$, ... satisfy the equations

$$\begin{gathered} f_0''' + \frac{1}{2} f_0 f_0'' = 0 \\ g_1''' + \frac{1}{2} f_0 g_1'' - f_0' g_1' + \frac{3}{2} f_0'' g_1 = -1 + f_0'^2 - \frac{1}{2} f_0 f_0'' \\ h_1''' + \frac{1}{2} f_0 h_1'' - f_0' h_1' + \frac{5}{2} f_0'' h_1 = \\ = -\frac{3}{2} g_1 g_1'' + g_1'^2 + 2 f_0' g_1' - \frac{1}{2} (f_0 g_1'' + f_0'' g_1) + 1 - f_0'^2 + \frac{1}{2} f_0 f_0'' \end{gathered} \tag{9}$$

with the boundary conditions

$$\begin{array}{llll} f_0 = 0, & f_0' = 0, & g_1 = 0, & g_1' = 0, \ldots \text{ when } \eta = 0 \\ f_0' \to 1, & g_1' \to 0, & & h_1' \to 0, \ldots \text{ when } \eta \to \infty \end{array} \tag{10}$$

If we assume that p_1, p_2, and p_3 are small enough, it is possible to expand ax, bx^2, and cx^2 into series of the form (6) by using (7). Substituting expressions for them into (8), collecting the terms with the same p_1, p_2, p_3 combinations, and comparing with (6) we finally find f_{ik}. For shearing stress at a wall $\tau = \mu\, \partial^2\psi/\partial y^2$ when $y = 0$ we have

$$\begin{gathered} \rho^{-1}\left(\frac{\nu U^3}{x}\right)^{-\frac{1}{2}} \tau = f''_{00}(0) + \delta p_1 f''_{10}(0) + \delta^2 (p_1^2 f''_{20}(0) + p_2 f''_{21}(0)) + \\ + \delta^3 (p_1^3 f''_{30}(0) + p_1 p_2 f''_{31}(0) + p_3 f''_{32}(0)) + \delta^4 (p_1^4 f''_{40}(0) + p_1^2 p_2 f''_{41}(0) + \\ + p_1 p_3 f''_{42}(0) + p_2^2 f''_{43}(0)) + \delta^5 (p_1^5 f''_{50}(0) + p_1^3 p_2 f''_{51}(0) + p_1^2 p_3 f''_{52}(0) + \\ + p_1 p_2^2 f''_{53}(0) + p_2 p_3 f''_{54}(0)) \end{gathered} \tag{11}$$

where the second derivatives of f_{ik}'' have the following values:

$$\begin{array}{lll} f''_{00}(0) = 0.33206, & f''_{10}(0) = 0.19290, & f''_{20}(0) = -0.03129 \\ f''_{21}(0) = -0.00318, & f''_{30}(0) = 0.01244, & f''_{31}(0) = 0.00132 \\ f''_{32}(0) = 0.00008, & f''_{40}(0) = -0.00679, & f''_{41}(0) = -0.00074 \\ f''_{42}(0) = -0.00003, & f''_{43}(0) = -0.00002, & f''_{50}(0) = 0.00423 \\ f''_{51}(0) = 0.00053, & f''_{52}(0) = 0.00002, & f''_{53}(0) = 0.00002 \\ f''_{54}(0) = 0.000001 & & \end{array}$$

The calculations from formula (11) coincide well with the results shown in Görtler and Witting [4] and Terrill [5]. To exemplify, breakaway of the boundary layer $U = 1-x^2$ takes place when $x = 0.64$, closely matching the value of $x = 0.67$ [5].

In order to examine the boundary layer on bodies with blunted foreparts we will introduce

$$x_1 = \frac{U'x}{U} - 1, \quad x_2 = \frac{U''x^2}{U} - 3\left(\frac{U'x}{U} - 1\right), \quad x_3 = \frac{U'''x^3}{U} - 6\frac{U''x^2}{U} + 15\left(\frac{U'x}{U} - 1\right) \tag{12}$$

Let $U = x\,(1 + ax^2 + bx^4 + cx^6)$; then

$$\begin{gathered} ax^2 = \frac{1}{D}\left(\frac{1}{2}x_1 - \frac{1}{4}x_2 + \frac{1}{16}x_3\right), \quad bx^4 = \frac{1}{D}\left(\frac{1}{8}x_2 - \frac{1}{16}x_3\right) \\ cx^6 = \frac{1}{D}\left(\frac{1}{48}x_3\right), \quad D = 1 - \frac{1}{2}x_1 + \frac{1}{8}x_2 - \frac{1}{48}x_3 \end{gathered} \tag{13}$$

Using the solutions found by expanding with respect to <u>x</u> (Curle [2]), we will by the above-described method obtain

$$f = F_{00} + x_1F_{10} + x_1^2F_{20} + x_2F_{21} + \ldots \tag{14}$$

and for shearing stress at a wall

$$\begin{gathered} \rho^{-1}\left(\frac{\nu U^3}{x}\right)^{-\frac{1}{2}}\tau = F''_{00}(0) + x_1F''_{10}(0) + x_1^2F''_{20}(0) + x_2F''_{21}(0) + \\ + x_1^3F''_{30}(0) + x_1x_2F''_{31}(0) + x_3F''_{32}(0) + x_1^4F''_{40}(0) + x_1^2x_2F''_{41}(0) + \\ + x_1x_3F''_{42}(0) + x_2^2F''_{43}(0) + x_1^5F''_{50}(0) + x_1^3x_2F''_{51}(0) + \\ + x_1^2x_3F''_{52}(0) + x_1x_2^2F''_{53}(0) + x_2x_3F''_{54}(0) \end{gathered} \tag{15}$$

Here

$$\begin{array}{lll} F''_{00}(0) = 1.232588, & F''_{10}(0) = 0.52145, & F''_{20}(0) = -0.06739 \\ F''_{21}(0) = -0.01731, & F''_{30}(0) = 0.01668, & F''_{31}(0) = 0.00758 \\ F''_{32}(0) = 0.00111, & F''_{40}(0) = -0.00627, & F''_{41}(0) = -0.00229 \\ F''_{42}(0) = -0.00058, & F''_{43}(0) = -0.00041, & F''_{50}(0) = 0.00276 \\ F''_{51}(0) = 0.00107, & F''_{52}(0) = 0.00016, & F''_{53}(0) = 0.00018 \\ F''_{54}(0) = 0.00008 & & \end{array}$$

By using (15) the shear stress and point of breakaway are calculated for various cases described in Curle [2] and Terrill [6]; in doing so good coincidence is ascertained. By way of example we cite below values of the magnitudes

$$T=\left(\frac{\nu x}{U^{3}}\right)^{\frac{1}{2}}\frac{\partial^{2}\psi}{\partial y^{2}}$$

calculated for $U = U_0\,(x-x^3 + 0.0789\,x^5)$ according to formula (15) and also borrowed from Curle [2].

$x =$	0.2	0.4	0.56	0.64	0.665	0.690	
$T =$	1.189	1.027	0.715	0.378	0	—	by formula (15)
$T =$	1.189	1.027	0.712	0.369	—	0	from Curle [2]

Received January 24, 1962

REFERENCES

1. N. J. Kotschin, I. A. Kibel, N. W. Rose. Theoretic Hydromechanics. Vol. 2, Gostekhizdat, 1948.

2. N. Curle. Accurate Solutions of the Laminar — Boundary-Layer Equations, For Flows Having a Stagnation Point and Separation. Aeronaut. Res. Concil Repts and Mem. N 3164 p. 11, 1960.

3. V. Ya. Shadov. Integrating Boundary Layer Equations. DAN SSSR, 126, 4, 730-732, 1959.

4. H. Görtler. A New Series for the Calculation of Steady Laminar Boundary Layer Flows J. Math and Mech. V. 6, p. 1, 1957.

5. H. Görtler. Witting H. Zu den Tanischen Grenzschichten, Oster, Ingr. Arch. V. 11, p. 111, 1957.

6. R. M. Terrill. Laminar Boundary-Layer Flow Near Separation With and Without Suction. Phylos, Trans, Roy. Soc. London, A253, N 1022, pp. 55-100, 1960.

DISTRIBUTION LIST

DEPARTMENT OF DEFENSE	Nr. Copies	MAJOR AIR COMMANDS	Nr. Copies
		AFSC	
		SCFDD	1
		DDC	25
HEADQUARTERS USAF		TDBTL	5
		TDBDP	6
AFCIN-3D2	1	AEDC (AEY)	1
ARL (ARB)	1	SSD (SSF)	2
		AFFTC (FTY)	1
		AFSWC (SWF)	1
OTHER AGENCIES		ASD (ASYIM)	1
CIA	1		
NSA	6		
DIA	9		
AID	2		
OTS	2		
AEC	2		
PWS	1		
NASA	1		
ARMY (FSTC)	3		
NAVY	3		
NAFEC	1		
RAND	1		